Depth versus Rigidity in the Design of International Trade Agreements

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1 Alternative Punishment Mechanisms

The one-period utility functions of the home and foreign government—W and W^*, respectively—are as follows:

$$\begin{aligned} W(t,\tau,a) &= a\,u(t) - t - u(\tau) \\ W^*(t,\tau,\alpha) &= \alpha\,u(\tau) - \tau - u(t) \end{aligned}$$

Losses are:

$$\begin{aligned} L(\tau) &= W(t,t_B,a) - W(t,\tau,a) = u(\tau) - u(\tau_B) \\ L^*(t) &= W^*(t_B,\tau,\alpha) - W^*(t,\tau,\alpha) = u(t) - u(t_B) \end{aligned}$$

Let χ_P denote the continuation payoff from the punishment that occurs if at least one player defects. Assume that χ_P is not a function of the specific value of the defection tariff.

Let χ_C denote the continuation payoff if the treaty remains in effect (neither player defects).

Recall that $a, \alpha \sim_{iid} U[1,A]$ for large A, $u' > 0$, and $u'' < 0$.

1.1 Optimal Tariffs

The home country's expected utility from violating the binding and not paying the fine (defection) is:

$$EU(D|t,a) = a\,u(t) - t - \int_1^A u(\tau(\alpha))\,dH(\alpha) + \int_{\alpha_S}^{\alpha_D} \sigma L(\tau(\alpha))\,dH(\alpha) + \delta\chi_P$$

So the optimal defection tariff solves:

$$\begin{aligned} \frac{\partial EU(D|t,a)}{\partial t} &= a\,u'(t) - 1 = 0 \\ &\Leftrightarrow \quad u'(t) = \frac{1}{a} \quad \Leftrightarrow \quad t_D(a) = u'^{-1}\left(\frac{1}{a}\right) \end{aligned}$$

This violates the binding iff:

$$t_D(a) = u'^{-1}\left(\frac{1}{a}\right) > t_B \quad \Leftrightarrow \quad \frac{1}{a} < u'(t_B) \quad \Leftrightarrow \quad a > \frac{1}{u'(t_B)} \equiv a_B$$

The home country's expected utility from violating the binding and paying the fine (settlement) is:

$$\begin{aligned} EU(S|t,a) &= a\,u(t) - t - \sigma L^*(t) - \int_1^A u(\tau(\alpha))dH(\alpha) + \int_{\alpha_S}^{\alpha_D} \sigma L(\tau(\alpha))dH(\alpha) \\ &\quad + H\left(\alpha_D\right)\delta\beta\chi_C + \left[1 - H\left(\alpha_D\right)\right]\delta\chi_P \end{aligned}$$

So the optimal settlement tariff solves:

$$\begin{aligned} \frac{\partial EU(S|t,a)}{\partial t} &= a\,u'(t) - 1 - \sigma u'(t) = 0 \\ &\Leftrightarrow \quad u'(t) = \frac{1}{a-\sigma} \quad \Leftrightarrow \quad t_S(a) = u'^{-1}\left(\frac{1}{a-\sigma}\right) \end{aligned}$$

This violates the binding iff:

$$\begin{aligned} t_S(a) &= u'^{-1}\left(\frac{1}{a-\sigma}\right) > t_B \quad \Leftrightarrow \quad \frac{1}{a-\sigma} < u'\left(t_B\right) \\ &\Leftrightarrow \quad a > \frac{1}{u'\left(t_B\right)} + \sigma \equiv a_S \end{aligned}$$

Note that: $t_S(a) < t_D(a)$ for all a.

The optimal cooperative tariff is:

$$t_B(a) = \begin{cases} t_D(a) & \text{if } a < a_B \\ t_B & \text{if } a_B \leq a \end{cases}$$

1.2 Equilibrium Regions

The home country's expected utility from actions C, S, and D given optimal tariff levels are:

$$\begin{aligned} EU\left(C|t_B(a),a\right) &= a\,u\left(t_B(a)\right) - t_B(a) - \int_1^A u(\tau(\alpha))dH(\alpha) + \int_{\alpha_S}^{\alpha_D} \sigma L(\tau(\alpha))dH(\alpha) \\ &\quad + H\left(\alpha_D\right)\delta\beta\chi_C + \left[1 - H\left(\alpha_D\right)\right]\delta\chi_P \\ EU\left(S|t_S(a),a\right) &= a\,u\left(t_S(a)\right) - t_S(a) - \sigma L^*\left(t_S(a)\right) - \int_1^A u(\tau(\alpha))dH(\alpha) + \int_{\alpha_S}^{\alpha_D} \sigma L(\tau(\alpha))dH(\alpha) \\ &\quad + H\left(\alpha_D\right)\delta\beta\chi_C + \left[1 - H\left(\alpha_D\right)\right]\delta\chi_P \\ EU\left(D|t_D(a),a\right) &= a\,u\left(t_D(a)\right) - t_D(a) - \int_1^A u(\tau(\alpha))dH(\alpha) + \int_{\alpha_S}^{\alpha_D} \sigma L(\tau(\alpha))dH(\alpha) + \delta\chi_P \end{aligned}$$

To compare utility from actions C and S, define for $a_S \leq a$:

$$\begin{aligned} \hat{\Delta}(a) &= EU\left(C|t_B(a),a\right) - EU\left(S|t_S(a),a\right) \\ &= a\,u\left(t_B\right) - t_B - a\,u\left(t_S(a)\right) + t_S(a) + \sigma L^*\left(t_S(a)\right) \end{aligned}$$

Note that $t_S\left(a_S\right) = t_B$, so $\hat{\Delta}\left(a_S\right) = 0$. Also:

$$\begin{aligned} \frac{\partial\hat{\Delta}}{\partial a} &= u\left(t_B\right) - u\left(t_S(a)\right) - (a-\sigma)\,u'\left(t_S(a)\right)\frac{\partial t_S(a)}{\partial a} + \frac{\partial t_S(a)}{\partial a} \\ &= u\left(t_B\right) - u\left(t_S(a)\right) < 0 \end{aligned}$$

So S strictly dominates C for all $a_S < a$.

To compare utility from actions S and D, define for $a_S \leq a$:

$$\begin{aligned}
\bar{\Delta}(a) &= EU\left(S|t_S(a),a\right) - EU\left(D|t_D(a),a\right) \\
&= a\,u\left(t_S(a)\right) - t_S(a) - \sigma L^*\left(t_S(a)\right) \\
&\quad -a\,u\left(t_D(a)\right) + t_D(a) + \delta H\left(\alpha_D\right)\left(\beta\chi_C - \chi_P\right) \\
\text{So:}\quad \frac{\partial\bar{\Delta}}{\partial a} &= \left(a-\sigma\right)u'\left(t_S(a)\right)\frac{\partial t_S(a)}{\partial a} - \frac{\partial t_S(a)}{\partial a} + u\left(t_S(a)\right) \\
&\quad + \frac{\partial t_D(a)}{\partial a} - au'\left(t_D(a)\right)\frac{\partial t_D(a)}{\partial a} - u\left(t_D(a)\right) \\
&= u\left(t_S(a)\right) - u\left(t_D(a)\right) < 0
\end{aligned}$$

So D strictly dominates S for sufficiently large values of a. By symmetry, indifference point a_D is implicitly defined by:

$$\begin{aligned}
\lambda &= a_D\left[u\left(t_S\left(a_D\right)\right) - u\left(t_D\left(a_D\right)\right)\right] + t_D\left(a_D\right) - t_S\left(a_D\right) \\
&\quad -\sigma L^*\left(t_S\left(a_D\right)\right) + \delta H\left(a_D\right)\left(\beta\chi_C - \chi_P\right) = 0
\end{aligned}$$

The equilibrium exists iff: $\overline{\Delta}\left(a_S\right) > 0$.

1.3 Continuation Values

Let $t_E(a)$ denote equilibrium tariffs when the institution is in place.

The continuation payoff for home from the treaty being in effect is:

$$\begin{aligned}
\chi_C &= \int_0^A \left[au\left(t_E(a)\right) - t_E(a)\right]dH(a) - \sigma\int_{a_S}^{a_D} L^*\left(t_E(a)\right)dH(a) - \int_1^A u\left(\tau_E(\alpha)\right)dH(\alpha) \\
&\quad +\sigma\int_{\alpha_S}^{\alpha_D} L^*\left(\tau_E(\alpha)\right)dH(\alpha) + \delta H\left(a_D\right)^2\beta\chi_C + \delta\left[1 - H\left(a_D\right)^2\right]\chi_P \\
&= \frac{\Psi}{1 - \delta\beta H\left(a_D\right)^2} \\
\text{where}\quad \Psi &= \int_1^A \left[\left(a-1\right)u\left(t_E(a)\right) - t_E(a)\right]dH(a) + \delta\left[1 - H\left(a_D\right)^2\right]\chi_P
\end{aligned}$$

1.4 Comparative Statics

Full Compliance

Recall that the binding is not violated if $a < a_S = \frac{1}{u'(t_B)} + \sigma$. So the probability that the binding is not violated is $H\left(a_S\right)$.

$$\frac{\partial a_S}{\partial t_B} = \frac{-u''\left(t_B\right)}{\left[u'\left(t_B\right)\right]^2} > 0 \quad \text{and} \quad \frac{\partial a_S}{\partial \sigma} = 1 > 0$$

Stability

The institution is stable if $a < a_D$. By the implicit function theorem:

$$\frac{\partial a_D}{\partial t_B} = -\frac{\lambda_{t_B}}{\lambda_{a_D}} \quad \text{and} \quad \frac{\partial a_D}{\partial \sigma} = -\frac{\lambda_\sigma}{\lambda_{a_D}}$$

Then for large A:

$$\begin{aligned}
\lambda_{a_D} &= (a_D - \sigma)\, u'(t_S(a_D)) \frac{\partial t_S(a_D)}{\partial a_D} - \frac{\partial t_S(a_D)}{\partial a_D} - a_D\, u'(t_D(a_D)) \frac{\partial t_D(a_D)}{\partial a_D} + \frac{\partial t_D(a_D)}{\partial a_D} \\
&\quad + u(t_S(a_D)) - u(t_D(a_D)) + \delta H(a_D) \frac{\partial(\beta\chi_C - \chi_P)}{\partial a_D} + \delta h(a_D)(\beta\chi_C - \chi_P) \\
&= u(t_S(a_D)) - u(t_D(a_D)) + \delta H(a_D) \frac{\partial(\beta\chi_C - \chi_P)}{\partial a_D} + \delta h(a_D)(\beta\chi_C - \chi_P) < 0 \\
\lambda_{t_B} &= \sigma u'(t_B) + \delta H(a_D) \frac{\partial(\beta\chi_C - \chi_P)}{\partial t_B} > 0 \\
\lambda_\sigma &= (a_D - \sigma)\, u'(t_S(a_D)) \frac{\partial t_S(a_D)}{\partial \sigma} - \frac{\partial t_S(a_D)}{\partial \sigma} \\
&\quad - [u(t_S(a_D)) - u(t_B)] + \delta H(a_D) \frac{\partial(\beta\chi_C - \chi_P)}{\partial \sigma} \\
&= -[u(t_S(a_D)) - u(t_B)] + \delta H(a_D) \frac{\partial(\beta\chi_C - \chi_P)}{\partial \sigma} < 0
\end{aligned}$$

$$\text{So:} \quad \frac{\partial a_D}{\partial t_B} > 0 \quad \text{and} \quad \frac{\partial a_D}{\partial \sigma} < 0$$

Depth versus Rigidity

Recall that χ_C is the expected utility of a state from being a member of the cooperative regime. In equilibrium, $\lambda = 0$. So for any pair (t_B, σ):

$$\chi_C = \frac{a_D\, [u(t_D(a_D)) - u(t_S(a_D))] - t_D(a_D) + t_S(a_D) + \sigma L^*(t_S(a_D))}{\delta\beta H(a_D)} + \frac{\chi_P}{\beta}$$

The two first-order conditions on the optimal pair (t_B, σ) are:

$$\begin{aligned}
\frac{d\chi_C}{dt_B} &= \frac{\partial \chi_C}{\partial t_B} + \frac{\partial \chi_C}{\partial a_D}\frac{\partial a_D}{\partial t_B} = 0 \\
\frac{d\chi_C}{d\sigma} &= \frac{\partial \chi_C}{\partial \sigma} + \frac{\partial \chi_C}{\partial a_D}\frac{\partial a_D}{\partial \sigma} = 0
\end{aligned}$$

This implies that:

$$\frac{\frac{\partial \chi_C}{\partial t_B}}{\frac{\partial \chi_C}{\partial \sigma}} = \frac{\frac{\partial a_D}{\partial t_B}}{\frac{\partial a_D}{\partial \sigma}}$$

$$\frac{\left(\frac{\partial \chi_C}{\partial t_B}\right)\left(\frac{\partial a_D}{\partial \sigma}\right)}{\frac{\partial \chi_C}{\partial \sigma}} = \frac{\partial a_D}{\partial t_B}$$

So for any pair (t_B, σ) that generates $\chi_C(t_B, \sigma) = \chi^*$:

$$
\begin{aligned}
\frac{dt_B}{d\sigma} &= \frac{-\left(\frac{d\chi_C}{d\sigma}\right)}{\frac{d\chi_C}{dt_B}} = \frac{-\left(\frac{\partial\chi_C}{\partial\sigma}+\frac{\partial\chi_C}{\partial a_D}\frac{\partial a_D}{\partial\sigma}\right)}{\frac{\partial\chi_C}{\partial t_B}+\frac{\partial\chi_C}{\partial a_D}\frac{\partial a_D}{\partial t_B}} \\
&= \frac{-\left(\frac{\partial\chi_C}{\partial\sigma}+\frac{\partial\chi_C}{\partial a_D}\frac{\partial a_D}{\partial\sigma}\right)}{\frac{\partial\chi_C}{\partial t_B}+\left(\frac{\partial\chi_C}{\partial a_D}\right)\frac{\left(\frac{\partial\chi_C}{\partial t_B}\right)\left(\frac{\partial a_D}{\partial\sigma}\right)}{\frac{\partial\chi_C}{\partial\sigma}}} \\
&= \frac{-\frac{\partial\chi_C}{\partial\sigma}\left(\frac{\partial\chi_C}{\partial\sigma}+\frac{\partial\chi_C}{\partial a_D}\frac{\partial a_D}{\partial\sigma}\right)}{\left(\frac{\partial\chi_C}{\partial t_B}\right)\left(\frac{\partial\chi_C}{\partial\sigma}\right)+\left(\frac{\partial\chi_C}{\partial a_D}\right)\left(\frac{\partial\chi_C}{\partial t_B}\right)\left(\frac{\partial a_D}{\partial\sigma}\right)} \\
&= \frac{-\frac{\partial\chi_C}{\partial\sigma}\left(\frac{\partial\chi_C}{\partial\sigma}+\frac{\partial\chi_C}{\partial a_D}\frac{\partial a_D}{\partial\sigma}\right)}{\frac{\partial\chi_C}{\partial t_B}\left[\frac{\partial\chi_C}{\partial\sigma}+\left(\frac{\partial\chi_C}{\partial a_D}\right)\left(\frac{\partial a_D}{\partial\sigma}\right)\right]} = \frac{-\frac{\partial\chi_C}{\partial\sigma}}{\frac{\partial\chi_C}{\partial t_B}}
\end{aligned}
$$

where:

$$
\begin{aligned}
\frac{\partial\chi_C}{\partial\sigma} &= \frac{1}{\delta\beta H(a_D)}\left[-a_D\,u'(t_S(a_D))\frac{\partial t_S(a_D)}{\partial\sigma}+\frac{\partial t_S(a_D)}{\partial\sigma}+\sigma\,u'(t_S(a_D))\frac{\partial t_S(a_D)}{\partial\sigma}\right] \\
&\quad+\frac{1}{\delta\beta H(a_D)}\left[L^*(t_S(a_D))\right]+\left(\frac{1}{\beta}\right)\frac{\partial\chi_P}{\partial\sigma} \\
&= \frac{L^*(t_S(a_D))}{\delta\beta H(a_D)}+\left(\frac{1}{\beta}\right)\frac{\partial\chi_P}{\partial\sigma} \\
\frac{\partial\chi_C}{\partial t_B} &= \frac{-\sigma u'(t_B)}{\delta\beta H(a_D)}+\left(\frac{1}{\beta}\right)\frac{\partial\chi_P}{\partial t_B}
\end{aligned}
$$

So:

$$
\begin{aligned}
\frac{dt_B}{d\sigma} &= \frac{-\frac{\partial\chi_C}{\partial\sigma}}{\frac{\partial\chi_C}{\partial t_B}} = \frac{-\left[\frac{L^*(t_S(a_D))}{\delta\beta H(a_D)}+\left(\frac{1}{\beta}\right)\frac{\partial\chi_P}{\partial\sigma}\right]}{\frac{-\sigma u'(t_B)}{\delta\beta H(a_D)}+\left(\frac{1}{\beta}\right)\frac{\partial\chi_P}{\partial t_B}} \\
&= \frac{L^*(t_S(a_D))+\delta H(a_D)\frac{\partial\chi_P}{\partial\sigma}}{\sigma u'(t_B)-\delta H(a_D)\frac{\partial\chi_P}{\partial t_B}} > 0 \quad \text{for small } A
\end{aligned}
$$

2 Asymmetric Type Distributions

Assume that home country type, a, is distributed according to distribution function $H(a)$. Denote home continuation payoffs by χ_N and χ_C.

Assume that foreign country type, α, is distributed according to distribution function $F(\alpha)$. Denote foreign continuation payoffs χ_N^* and χ_C^*.

The one-period utility functions of the home and foreign government—W and W^*, respectively—are as follows:

$$
\begin{aligned}
W(t,\tau,a) &= a\,u(t)-t-u(\tau) \\
W^*(t,\tau,\alpha) &= \alpha\,u(\tau)-\tau-u(t)
\end{aligned}
$$

Losses are:

$$\begin{aligned} L(\tau) &= W\left(t, \tau_B, a\right) - W(t, \tau, a) = u(\tau) - u\left(\tau_B\right) \\ L^*(t) &= W^*\left(t_B, \tau, \alpha\right) - W^*(t, \tau, \alpha) = u(t) - u\left(t_B\right) \end{aligned}$$

2.1 Optimal Tariffs

Home

The home country's expected utility from from violating the binding and not paying compensation (defection) is:

$$U(D|t,a) = a\,u(t) - t - \int u(\tau(\alpha))\,dF(\alpha) + \int_{\alpha_S}^{\alpha_D} \sigma L\left(\tau\left(\alpha\right)\right)\,dF(\alpha) + \delta\chi_N$$

So the optimal defection tariff solves:

$$\begin{aligned} \frac{\partial U(D|t,a)}{\partial t} &= a\,u'(t) - 1 = 0 \\ &\Leftrightarrow \quad u'(t) = \frac{1}{a} \quad \Leftrightarrow \quad t_D(a) = u'^{-1}\left(\frac{1}{a}\right) \end{aligned}$$

This violates the home binding iff:

$$t_D(a) = u'^{-1}\left(\frac{1}{a}\right) > t_B \quad \Leftrightarrow \quad \frac{1}{a} < u'\left(t_B\right) \quad \Leftrightarrow \quad a > \frac{1}{u'\left(t_B\right)} \equiv a_B$$

The home country's expected utility from violating the binding and paying compensation (settlement) is:

$$\begin{aligned} U(S|t,a) &= a\,u(t) - t - \sigma L^*(t) - \int u(\tau(\alpha))\,dF(\alpha) + \int_{\alpha_S}^{\alpha_D} \sigma L(\tau(\alpha))\,dF(\alpha) \\ &\quad + F\left(\alpha_D\right)\delta\beta\chi_C + \left[1 - F\left(\alpha_D\right)\right]\delta\chi_N \end{aligned}$$

So the optimal settlement tariff solves:

$$\begin{aligned} \frac{\partial U(S|t,a)}{\partial t} &= a\,u'(t) - 1 - \sigma u'(t) = 0 \\ &\Leftrightarrow \quad u'(t) = \frac{1}{a-\sigma} \quad \Leftrightarrow \quad t_S(a) = u'^{-1}\left(\frac{1}{a-\sigma}\right) \end{aligned}$$

This violates the home binding iff:

$$\begin{aligned} t_S(a) &= u'^{-1}\left(\frac{1}{a-\sigma}\right) > t_B \quad \Leftrightarrow \quad \frac{1}{a-\sigma} < u'\left(t_B\right) \\ &\Leftrightarrow \quad a > \frac{1}{u'\left(t_B\right)} + \sigma \equiv a_S \end{aligned}$$

Note that: $t_S(a) < t_D(a)$ for all a. The optimal cooperative tariff is:

$$t_B(a) = \begin{cases} t_D(a) & \text{if } a < a_B \\ t_B & \text{if } a_B \leq a \end{cases}$$

Foreign

The foreign country's expected utility from from violating the binding and not paying compensation (defection) is:

$$U^*(D|\tau,\alpha) \quad = \quad \alpha\, u(\tau) - \tau - \int u(t(a))\, dH(a) + \int_{a_S}^{a_D} \sigma L^* \left(t\left(a\right)\right)\, dH(a) + \delta\chi_N^*$$

So the optimal defection tariff solves:

$$\begin{aligned} \frac{\partial U^*(D|\tau,\alpha)}{\partial\tau} &= \alpha\, u'(\tau) - 1 = 0 \\ &\Leftrightarrow \quad u'(\tau) = \frac{1}{\alpha} \quad \Leftrightarrow \quad \tau_D(\alpha) = u'^{-1}\left(\frac{1}{\alpha}\right) \end{aligned}$$

This violates the foreign binding iff:

$$\tau_D(\alpha) = u'^{-1}\left(\frac{1}{\alpha}\right) > \tau_B \quad \Leftrightarrow \quad \frac{1}{\alpha} < u'\left(\tau_B\right) \quad \Leftrightarrow \quad \alpha > \frac{1}{u'\left(\tau_B\right)} \equiv \alpha_B$$

The foreign country's expected utility from violating the binding and paying compensation (settlement) is:

$$\begin{aligned} U^*(S|\tau,\alpha) \quad &= \quad \alpha\, u(\tau) - \tau - \sigma L(\tau) - \int u(t(a))\, dH(a) + \int_{a_S}^{a_D} \sigma L^*(t(a))\, dH(a) \\ &\quad + H\left(a_D\right)\delta\beta\chi_C^* + \left[1 - H\left(a_D\right)\right]\delta\chi_N^* \end{aligned}$$

So the optimal settlement tariff solves:

$$\begin{aligned} \frac{\partial U^*(S|\tau,\alpha)}{\partial\tau} &= \alpha\, u'(\tau) - 1 - \sigma u'(\tau) = 0 \\ &\Leftrightarrow \quad u'(\tau) = \frac{1}{\alpha - \sigma} \quad \Leftrightarrow \quad \tau_S(\alpha) = u'^{-1}\left(\frac{1}{\alpha - \sigma}\right) \end{aligned}$$

This violates the foreign binding iff:

$$\begin{aligned} \tau_S(\alpha) &= u'^{-1}\left(\frac{1}{\alpha - \sigma}\right) > \tau_B \quad \Leftrightarrow \quad \frac{1}{\alpha - \sigma} < u'\left(\tau_B\right) \\ &\Leftrightarrow \quad \alpha > \frac{1}{u'\left(\tau_B\right)} + \sigma \equiv \alpha_S \end{aligned}$$

Note that: $\tau_S(\alpha) < \tau_D(\alpha)$ for all α. The optimal cooperative tariff is:

$$\tau_B(\alpha) = \begin{cases} \tau_D(\alpha) & \text{if } \alpha < \alpha_B \\ \tau_B & \text{if } \alpha_B \leq \alpha \end{cases}$$

2.2 Equilibrium Regions

Home

The home country's expected utility from actions C, S, and D given tariff levels above are:

$$\begin{aligned} U\left(C|t_B(a),a\right) &= a\,u\left(t_B(a)\right)-t_B(a)-\int u(\tau(\alpha))dF(\alpha)+\int_{\alpha_S}^{\alpha_D}\sigma L(\tau(\alpha))dF(\alpha) \\ &\quad +F\left(\alpha_D\right)\delta\beta\chi_C+\left[1-F\left(\alpha_D\right)\right]\delta\chi_N \\ U\left(S|t_S(a),a\right) &= a\,u\left(t_S(a)\right)-t_S(a)-\sigma L^*\left(t_S(a)\right)-\int u(\tau(\alpha))dF(\alpha)+\int_{\alpha_S}^{\alpha_D}\sigma L(\tau(\alpha))dF(\alpha) \\ &\quad +F\left(\alpha_D\right)\delta\beta\chi_C+\left[1-F\left(\alpha_D\right)\right]\delta\chi_N \\ U\left(D|t_D(a),a\right) &= a\,u\left(t_D(a)\right)-t_D(a)-\int u(\tau(\alpha))dF(\alpha)+\int_{\alpha_S}^{\alpha_D}\sigma L(\tau(\alpha))dF(\alpha)+\delta\chi_N \end{aligned}$$

To compare home utility from actions C and S, define for $a_S \leq a$:

$$\begin{aligned} \hat{\Delta}(a) &= U\left(C|t_B(a),a\right)-U\left(S|t_S(a),a\right) \\ &= a\,u\left(t_B\right)-t_B-a\,u\left(t_S(a)\right)+t_S(a)+\sigma L^*\left(t_S(a)\right) \end{aligned}$$

Note that $t_S\left(a_S\right)=t_B$, so $\hat{\Delta}\left(a_S\right)=0$. Also:

$$\begin{aligned} \frac{\partial\hat{\Delta}}{\partial a} &= u\left(t_B\right)-u\left(t_S(a)\right)-(a-\sigma)\,u'\left(t_S(a)\right)\frac{\partial t_S(a)}{\partial a}+\frac{\partial t_S(a)}{\partial a} \\ &= u\left(t_B\right)-u\left(t_S(a)\right)<0 \end{aligned}$$

So S strictly dominates C for all $a_S < a$.

To compare home utility from actions S and D, define for $a_S \leq a$:

$$\begin{aligned} \bar{\Delta}(a) &= U\left(S|t_S(a),a\right)-U\left(D|t_D(a),a\right) \\ &= a\,u\left(t_S(a)\right)-t_S(a)-\sigma L^*\left(t_S(a)\right) \\ &\quad -a\,u\left(t_D(a)\right)+t_D(a)+\delta F\left(\alpha_D\right)\left(\beta\chi_C-\chi_N\right) \\ \text{So:}\quad \frac{\partial\bar{\Delta}}{\partial a} &= (a-\sigma)\,u'\left(t_S(a)\right)\frac{\partial t_S(a)}{\partial a}-\frac{\partial t_S(a)}{\partial a}+u\left(t_S(a)\right) \\ &\quad +\frac{\partial t_D(a)}{\partial a}-au'\left(t_D(a)\right)\frac{\partial t_D(a)}{\partial a}-u\left(t_D(a)\right) \\ &= u\left(t_S(a)\right)-u\left(t_D(a)\right)<0 \end{aligned}$$

So D strictly dominates S for sufficiently large values of a.

Indifference point a_D is implicitly defined by:

$$\begin{aligned} \lambda &= a_D\left[u\left(t_S\left(a_D\right)\right)-u\left(t_D\left(a_D\right)\right)\right]+t_D\left(a_D\right)-t_S\left(a_D\right) \\ &\quad -\sigma L^*\left(t_S\left(a_D\right)\right)+\delta F\left(\alpha_D\right)\left(\beta\chi_C-\chi_N\right)=0 \end{aligned}$$

The equilibrium exists iff: $\overline{\Delta}\left(a_S\right) > 0$.

Foreign

The foreign country's expected utility from actions C, S, and D given tariff levels above are:

$$\begin{aligned}
U^*\left(C|\tau_B(\alpha),\alpha\right) &= \alpha\, u\left(\tau_B(\alpha)\right) - \tau_B(\alpha) - \int u(t(a))dH(a) + \int_{a_S}^{a_D} \sigma L^*(t(a))dH(a) \\
&\quad + H\left(a_D\right)\delta\beta\chi_C^* + \left[1 - H\left(a_D\right)\right]\delta\chi_N^* \\
U^*\left(S|\tau_S(\alpha),\alpha\right) &= \alpha\, u\left(\tau_S(\alpha)\right) - \tau_S(\alpha) - \sigma L\left(\tau_S(\alpha)\right) - \int u(t(a))dH(a) + \int_{a_S}^{a_D} \sigma L^*(t(a))dH(a) \\
&\quad + H\left(a_D\right)\delta\beta\chi_C^* + \left[1 - H\left(a_D\right)\right]\delta\chi_N^* \\
U^*\left(D|\tau_D(\alpha),\alpha\right) &= \alpha\, u\left(\tau_D(\alpha)\right) - \tau_D(\alpha) - \int u(t(a))dH(a) + \int_{a_S}^{a_D} \sigma L^*(t(a))dH(a) + \delta\chi_N^*
\end{aligned}$$

To compare foreign utility from actions C and S, define for $\alpha_S \leq \alpha$:

$$\begin{aligned}
\hat{\Delta}^*(\alpha) &= U^*\left(C|\tau_B(\alpha),\alpha\right) - U^*\left(S|\tau_S(\alpha),\alpha\right) \\
&= \alpha\, u\left(\tau_B\right) - \tau_B - \alpha\, u\left(\tau_S(\alpha)\right) + \tau_S(\alpha) + \sigma L\left(\tau_S(\alpha)\right)
\end{aligned}$$

Note that $\tau_S\left(\alpha_S\right) = \tau_B$, so $\hat{\Delta}^*\left(\alpha_S\right) = 0$. Also:

$$\begin{aligned}
\frac{\partial\hat{\Delta}^*}{\partial\alpha} &= u\left(\tau_B\right) - u\left(\tau_S(\alpha)\right) - \left(\alpha - \sigma\right) u'\left(\tau_S(\alpha)\right)\frac{\partial\tau_S(\alpha)}{\partial\alpha} + \frac{\partial\tau_S(\alpha)}{\partial\alpha} \\
&= u\left(\tau_B\right) - u\left(\tau_S(\alpha)\right) < 0
\end{aligned}$$

So S strictly dominates C for all $\alpha_S < \alpha$.

To compare foreign utility from actions S and D, define for $\alpha_S \leq \alpha$:

$$\begin{aligned}
\bar{\Delta}^*(\alpha) &= U^*\left(S|\tau_S(\alpha),\alpha\right) - U^*\left(D|\tau_D(\alpha),\alpha\right) \\
&= \alpha\, u\left(\tau_S(\alpha)\right) - \tau_S(\alpha) - \sigma L\left(\tau_S(\alpha)\right) \\
&\quad - \alpha\, u\left(\tau_D(\alpha)\right) + \tau_D(\alpha) + \delta H\left(a_D\right)\left(\beta\chi_C^* - \chi_N^*\right) \\
\text{So:} \qquad \frac{\partial\bar{\Delta}^*}{\partial\alpha} &= \left(\alpha - \sigma\right) u'\left(\tau_S(\alpha)\right)\frac{\partial\tau_S(\alpha)}{\partial\alpha} - \frac{\partial\tau_S(\alpha)}{\partial\alpha} + u\left(\tau_S(\alpha)\right) \\
&\quad + \frac{\partial\tau_D(\alpha)}{\partial\alpha} - \alpha u'\left(\tau_D(\alpha)\right)\frac{\partial\tau_D(\alpha)}{\partial\alpha} - u\left(\tau_D(\alpha)\right) \\
&= u\left(\tau_S(\alpha)\right) - u\left(\tau_D(\alpha)\right) < 0
\end{aligned}$$

So D strictly dominates S for sufficiently large values of α.

Indifference point α_D is implicitly defined by:

$$\begin{aligned}\lambda^* &= \alpha_D \left[u\left(\tau_S\left(\alpha_D\right)\right) - u\left(\tau_D\left(\alpha_D\right)\right)\right] + \tau_D\left(\alpha_D\right) - \tau_S\left(\alpha_D\right) \\ &\quad -\sigma L\left(\tau_S\left(\alpha_D\right)\right) + \delta H\left(a_D\right)\left(\beta\chi_C^* - \chi_N^*\right) = 0\end{aligned}$$

The equilibrium exists iff: $\bar{\Delta}^*\left(\alpha_S\right) > 0$.

2.3 Continuation Values

Let $t_E(a)$ and $\tau_E(\alpha)$ denote equilibrium tariffs of the home and foreign country, respectively, when the institution is in place.

If the institution does not exist, then the home and foreign country choose $t_D(a)$ and $\tau_D(\alpha)$ in every time period. This yields anarchy continuation payoffs:

$$\begin{aligned}\chi_N &= \frac{1}{1-\delta}\left\{\int\left[a\,u\left(t_D(a)\right) - t_D(a)\right]dH(a) - \int u\left(\tau_D\left(\alpha\right)\right)dF\left(\alpha\right)\right\} \\ \chi_N^* &= \frac{1}{1-\delta}\left\{\int\left[\alpha\,u\left(\tau_D\left(\alpha\right)\right) - \tau_D\left(\alpha\right)\right]dF\left(\alpha\right) - \int u\left(t_D\left(a\right)\right)dH\left(a\right)\right\}\end{aligned}$$

Home

The continuation payoff for home from the treaty being in effect is:

$$\begin{aligned}\chi_C &= \int\left[au\left(t_E(a)\right) - t_E(a)\right]dH(a) - \sigma\int_{a_S}^{a_D} L^*\left(t_E(a)\right)dH(a) - \int u\left(\tau_E(\alpha)\right)dF(\alpha) \\ &\quad +\sigma\int_{\alpha_S}^{\alpha_D} L\left(\tau_E(\alpha)\right)dF(\alpha) + \delta H\left(a_D\right)F\left(\alpha_D\right)\beta\chi_C + \delta\left[1 - H\left(a_D\right)F\left(\alpha_D\right)\right]\chi_N \\ &= \frac{\Psi}{1 - \delta\beta H\left(a_D\right)F\left(\alpha_D\right)} \\ \text{where}\quad \Psi &= \int\left[au\left(t_E(a)\right) - t_E(a)\right]dH(a) - \sigma\int_{a_S}^{a_D} L^*\left(t_E(a)\right)dH(a) - \int u\left(\tau_E(\alpha)\right)dF(\alpha) \\ &\quad +\sigma\int_{\alpha_S}^{\alpha_D} L\left(\tau_E(\alpha)\right)dF(\alpha) + \delta\left[1 - H\left(a_D\right)F\left(\alpha_D\right)\right]\chi_N\end{aligned}$$

Foreign

The continuation payoff for foreign from the treaty being in effect is:

$$\begin{aligned}
\chi_C^* &= \int \left[\alpha\, u\left(\tau_E(\alpha)\right) - \tau_E(\alpha)\right] dF\left(\alpha\right) - \sigma \int_{\alpha_S}^{\alpha_D} L\left(\tau_E(\alpha)\right) dF(\alpha) - \int u\left(t_E(a)\right) dH(a) \\
&\quad + \sigma \int_{a_S}^{a_D} L^*\left(t_E(a)\right) dH(a) + \delta H\left(a_D\right) F\left(\alpha_D\right) \beta \chi_C^* + \delta\left[1 - H\left(a_D\right) F\left(\alpha_D\right)\right] \chi_N^* \\
&= \frac{\Psi^*}{1 - \delta\beta H\left(a_D\right) F\left(\alpha_D\right)} \\
\text{where} \quad \Psi^* &= \int \left[\alpha\, u\left(\tau_E(\alpha)\right) - \tau_E(\alpha)\right] dF\left(\alpha\right) - \sigma \int_{\alpha_S}^{\alpha_D} L\left(\tau_E(\alpha)\right) dF(\alpha) - \int u\left(t_E(a)\right) dH(a) \\
&\quad + \sigma \int_{a_S}^{a_D} L^*\left(t_E(a)\right) dH(a) + \delta\left[1 - H\left(a_D\right) F\left(\alpha_D\right)\right] \chi_N^*
\end{aligned}$$

2.4 Comparative Statics

Full Compliance

Recall that the home country does not violate its binding if $a \leq a_S = \frac{1}{u'(t_B)} + \sigma$.

$$\frac{\partial a_S}{\partial t_B} = \frac{-u''\left(t_B\right)}{\left[u'\left(t_B\right)\right]^2} > 0 \quad \text{and} \quad \frac{\partial a_S}{\partial \sigma} = 1 > 0$$

Recall that the foreign country does not violate its binding if $\alpha \leq \alpha_S = \frac{1}{u'(\tau_B)} + \sigma$.

$$\frac{\partial \alpha_S}{\partial \tau_B} = \frac{-u''\left(\tau_B\right)}{\left[u'\left(\tau_B\right)\right]^2} > 0 \quad \text{and} \quad \frac{\partial \alpha_S}{\partial \sigma} = 1 > 0$$

Stability

The cutpoints (a_D, α_D) are implicitly defined by the system of equations:

$$\begin{aligned}
\lambda\left(a_D, \alpha_D\right) &= 0 \\
\lambda^*\left(a_D, \alpha_D\right) &= 0
\end{aligned}$$

By Cramer's Rule:

$$\frac{\partial a_D}{\partial t_B} = \frac{\begin{vmatrix} -\lambda_{t_B} & \lambda_{\alpha_D} \\ -\lambda^*_{t_B} & \lambda^*_{\alpha_D} \end{vmatrix}}{\begin{vmatrix} \lambda_{a_D} & \lambda_{\alpha_D} \\ \lambda^*_{a_D} & \lambda^*_{\alpha_D} \end{vmatrix}} \quad \text{and} \quad \frac{\partial a_D}{\partial \sigma} = \frac{\begin{vmatrix} -\lambda_{\sigma} & \lambda_{\alpha_D} \\ -\lambda^*_{\sigma} & \lambda^*_{\alpha_D} \end{vmatrix}}{\begin{vmatrix} \lambda_{a_D} & \lambda_{\alpha_D} \\ \lambda^*_{a_D} & \lambda^*_{\alpha_D} \end{vmatrix}}$$

$$\frac{\partial \alpha_D}{\partial \tau_B} = \frac{\begin{vmatrix} -\lambda_{\tau_B} & \lambda_{a_D} \\ -\lambda^*_{\tau_B} & \lambda^*_{a_D} \end{vmatrix}}{\begin{vmatrix} \lambda_{\alpha_D} & \lambda_{a_D} \\ \lambda^*_{\alpha_D} & \lambda^*_{a_D} \end{vmatrix}} \quad \text{and} \quad \frac{\partial \alpha_D}{\partial \sigma} = \frac{\begin{vmatrix} -\lambda_{\sigma} & \lambda_{a_D} \\ -\lambda^*_{\sigma} & \lambda^*_{a_D} \end{vmatrix}}{\begin{vmatrix} \lambda_{\alpha_D} & \lambda_{a_D} \\ \lambda^*_{\alpha_D} & \lambda^*_{a_D} \end{vmatrix}}$$

where:

$$\begin{aligned}
\lambda_{a_D} &= u(t_S(a_D)) - u(t_D(a_D)) + \delta F(\alpha_D)\beta \frac{\partial \chi_C}{\partial a_D} \\
\lambda_{\alpha_D} &= \delta F(\alpha_D)\beta \left(\frac{\partial \chi_C}{\partial \alpha_D}\right) + \delta f(\alpha_D)(\beta\chi_C - \chi_N) \\
\lambda_{t_B} &= \sigma u'(t_B) + \delta F(\alpha_D)\beta \left(\frac{\partial \chi_C}{\partial t_B}\right) \\
\lambda_{\tau_B} &= \delta F(\alpha_D)\beta \left(\frac{\partial \chi_C}{\partial \tau_B}\right) \\
\lambda_{\sigma} &= -L^*(t_S(a_D)) + \delta F(\alpha_D)\beta \frac{\partial \chi_C}{\partial \sigma}
\end{aligned}$$

$$\begin{aligned}
\lambda^*_{a_D} &= \delta H(a_D)\beta \left(\frac{\partial \chi^*_C}{\partial a_D}\right) + \delta h(a_D)(\beta\chi^*_C - \chi^*_N) \\
\lambda^*_{\alpha_D} &= u(\tau_S(\alpha_D)) - u(\tau_D(\alpha_D)) + \delta H(a_D)\beta \frac{\partial \chi^*_C}{\partial \alpha_D} \\
\lambda^*_{t_B} &= \delta H(a_D)\beta \left(\frac{\partial \chi^*_C}{\partial t_B}\right) \\
\lambda^*_{\tau_B} &= \sigma u'(\tau_B) + \delta H(a_D)\beta \frac{\partial \chi^*_C}{\partial \tau_B} \\
\lambda^*_{\sigma} &= -L(\tau_S(\alpha_D)) + \delta H(a_D)\beta \frac{\partial \chi^*_C}{\partial \sigma}
\end{aligned}$$

Comparative statics on a_D

As $h(a_D)$, $H(a_D)$, $f(\alpha_D)$, $F(\alpha_D)$ grow small:

$$\begin{aligned}
\begin{vmatrix} \lambda_{a_D} & \lambda_{\alpha_D} \\ \lambda^*_{a_D} & \lambda^*_{\alpha_D} \end{vmatrix} &= \lambda_{a_D}\lambda^*_{\alpha_D} - \lambda_{\alpha_D}\lambda^*_{a_D} \\
&= \left[u(t_S(a_D)) - u(t_D(a_D)) + \delta F(\alpha_D)\beta \frac{\partial \chi_C}{\partial a_D}\right] \times \left[u(\tau_S(\alpha_D)) - u(\tau_D(\alpha_D)) + \delta H(a_D)\beta \frac{\partial \chi^*_C}{\partial \alpha_D}\right] \\
&\quad - \left[\delta F(\alpha_D)\beta \left(\frac{\partial \chi_C}{\partial \alpha_D}\right) + \delta f(\alpha_D)(\beta\chi_C - \chi_N)\right] \times \left[\delta H(a_D)\beta \left(\frac{\partial \chi^*_C}{\partial a_D}\right) + \delta h(a_D)(\beta\chi^*_C - \chi^*_N)\right] \\
&\to [u(t_S(a_D)) - u(t_D(a_D))] \times [u(\tau_S(\alpha_D)) - u(\tau_D(\alpha_D))] > 0
\end{aligned}$$

$$\begin{vmatrix} -\lambda_{t_B} & \lambda_{\alpha_D} \\ -\lambda^*_{t_B} & \lambda^*_{\alpha_D} \end{vmatrix} = -\lambda_{t_B}\lambda^*_{\alpha_D} + \lambda_{\alpha_D}\lambda^*_{t_B}$$

$$= -\left[\sigma u'(t_B) + \delta F(\alpha_D)\beta\left(\frac{\partial \chi_C}{\partial t_B}\right)\right] \times \left[u(\tau_S(\alpha_D)) - u(\tau_D(\alpha_D)) + \delta H(a_D)\beta\frac{\partial \chi^*_C}{\partial \alpha_D}\right]$$

$$+ \left[\delta F(\alpha_D)\beta\left(\frac{\partial \chi_C}{\partial \alpha_D}\right) + \delta f(\alpha_D)(\beta\chi_C - \chi_N)\right] \times \left[\delta H(a_D)\beta\left(\frac{\partial \chi^*_C}{\partial t_B}\right)\right]$$

$$\rightarrow -\sigma u'(t_B)\left[u(\tau_S(\alpha_D)) - u(\tau_D(\alpha_D))\right] > 0$$

$$\begin{vmatrix} -\lambda_{\sigma} & \lambda_{\alpha_D} \\ -\lambda^*_{\sigma} & \lambda^*_{\alpha_D} \end{vmatrix} = -\lambda_{\sigma}\lambda^*_{\alpha_D} + \lambda_{\alpha_D}\lambda^*_{\sigma}$$

$$= -\left[-L^*(t_S(a_D)) + \delta F(\alpha_D)\beta\frac{\partial \chi_C}{\partial \sigma}\right] \times \left[u(\tau_S(\alpha_D)) - u(\tau_D(\alpha_D)) + \delta H(a_D)\beta\frac{\partial \chi^*_C}{\partial \alpha_D}\right]$$

$$+ \left[\delta F(\alpha_D)\beta\left(\frac{\partial \chi_C}{\partial \alpha_D}\right) + \delta f(\alpha_D)(\beta\chi_C - \chi_N)\right] \times \left[-L(\tau_S(\alpha_D)) + \delta H(a_D)\beta\frac{\partial \chi^*_C}{\partial \sigma}\right]$$

$$\rightarrow L^*(t_S(a_D))\left[u(\tau_S(\alpha_D)) - u(\tau_D(\alpha_D))\right] < 0$$

So:

$$\frac{\partial a_D}{\partial t_B} = \frac{\begin{vmatrix} -\lambda_{t_B} & \lambda_{\alpha_D} \\ -\lambda^*_{t_B} & \lambda^*_{\alpha_D} \end{vmatrix}}{\begin{vmatrix} \lambda_{a_D} & \lambda_{\alpha_D} \\ \lambda^*_{a_D} & \lambda^*_{\alpha_D} \end{vmatrix}} \rightarrow \frac{-\sigma u'(t_B)\left[u(\tau_S(\alpha_D)) - u(\tau_D(\alpha_D))\right]}{\left[u(t_S(a_D)) - u(t_D(a_D))\right] \times \left[u(\tau_S(\alpha_D)) - u(\tau_D(\alpha_D))\right]} > 0$$

$$\frac{\partial a_D}{\partial \sigma} = \frac{\begin{vmatrix} -\lambda_{\sigma} & \lambda_{\alpha_D} \\ -\lambda^*_{\sigma} & \lambda^*_{\alpha_D} \end{vmatrix}}{\begin{vmatrix} \lambda_{a_D} & \lambda_{\alpha_D} \\ \lambda^*_{a_D} & \lambda^*_{\alpha_D} \end{vmatrix}} \rightarrow \frac{L^*(t_S(a_D))\left[u(\tau_S(\alpha_D)) - u(\tau_D(\alpha_D))\right]}{\left[u(t_S(a_D)) - u(t_D(a_D))\right] \times \left[u(\tau_S(\alpha_D)) - u(\tau_D(\alpha_D))\right]} < 0$$

Comparative Statics on α_D

As $h(a_D)$, $H(a_D)$, $f(\alpha_D)$, $F(\alpha_D)$ grow small:

$$\begin{vmatrix} \lambda_{\alpha_D} & \lambda_{a_D} \\ \lambda^*_{\alpha_D} & \lambda^*_{a_D} \end{vmatrix} = \lambda_{\alpha_D}\lambda^*_{a_D} - \lambda_{a_D}\lambda^*_{\alpha_D}$$

$$= \left[\delta F(\alpha_D)\beta\left(\frac{\partial \chi_C}{\partial \alpha_D}\right) + \delta f(\alpha_D)(\beta\chi_C - \chi_N)\right] \times \left[\delta H(a_D)\beta\left(\frac{\partial \chi^*_C}{\partial a_D}\right) + \delta h(a_D)(\beta\chi^*_C - \chi^*_N)\right]$$

$$- \left[u(t_S(a_D)) - u(t_D(a_D)) + \delta F(\alpha_D)\beta\frac{\partial \chi_C}{\partial a_D}\right] \times \left[u(\tau_S(\alpha_D)) - u(\tau_D(\alpha_D)) + \delta H(a_D)\beta\frac{\partial \chi^*_C}{\partial \alpha_D}\right]$$

$$\rightarrow -\left[u(t_S(a_D)) - u(t_D(a_D))\right] \times \left[u(\tau_S(\alpha_D)) - u(\tau_D(\alpha_D))\right] < 0$$

$$\begin{vmatrix} -\lambda_{\tau_B} & \lambda_{a_D} \\ -\lambda^*_{\tau_B} & \lambda^*_{a_D} \end{vmatrix} = -\lambda_{\tau_B}\lambda^*_{a_D} + \lambda_{a_D}\lambda^*_{\tau_B}$$

$$= -\delta F(\alpha_D)\,\beta\left(\frac{\partial \chi_C}{\partial \tau_B}\right) \times \left[\delta H(a_D)\,\beta\left(\frac{\partial \chi^*_C}{\partial a_D}\right) + \delta h(a_D)\left(\beta\chi^*_C - \chi^*_N\right)\right]$$

$$+ \left[u(t_S(a_D)) - u(t_D(a_D)) + \delta F(\alpha_D)\,\beta\frac{\partial \chi_C}{\partial a_D}\right] \times \left[\sigma u'(\tau_B) + \delta H(a_D)\,\beta\frac{\partial \chi^*_C}{\partial \tau_B}\right]$$

$$\to \sigma u'(\tau_B)\left[u(t_S(a_D)) - u(t_D(a_D))\right] < 0$$

$$\begin{vmatrix} -\lambda_{\sigma} & \lambda_{a_D} \\ -\lambda^*_{\sigma} & \lambda^*_{a_D} \end{vmatrix} = -\lambda_{\sigma}\lambda^*_{a_D} + \lambda_{a_D}\lambda^*_{\sigma}$$

$$= -\left[-L^*(t_S(a_D)) + \delta F(\alpha_D)\,\beta\frac{\partial \chi_C}{\partial \sigma}\right] \times \left[\delta H(a_D)\,\beta\left(\frac{\partial \chi^*_C}{\partial a_D}\right) + \delta h(a_D)\left(\beta\chi^*_C - \chi^*_N\right)\right]$$

$$+ \left[u(t_S(a_D)) - u(t_D(a_D)) + \delta F(\alpha_D)\,\beta\frac{\partial \chi_C}{\partial a_D}\right] \times \left[-L(\tau_S(\alpha_D)) + \delta H(a_D)\,\beta\frac{\partial \chi^*_C}{\partial \sigma}\right]$$

$$\to -L(\tau_S(\alpha_D))\left[u(t_S(a_D)) - u(t_D(a_D))\right] > 0$$

So:

$$\frac{\partial \alpha_D}{\partial \tau_B} = \frac{\begin{vmatrix} -\lambda_{\tau_B} & \lambda_{a_D} \\ -\lambda^*_{\tau_B} & \lambda^*_{a_D} \end{vmatrix}}{\begin{vmatrix} \lambda_{\alpha_D} & \lambda_{a_D} \\ \lambda^*_{\alpha_D} & \lambda^*_{a_D} \end{vmatrix}} \to \frac{\sigma u'(\tau_B)\left[u(t_S(a_D)) - u(t_D(a_D))\right]}{-\left[u(t_S(a_D)) - u(t_D(a_D))\right] \times \left[u(\tau_S(\alpha_D)) - u(\tau_D(\alpha_D))\right]} > 0$$

So:

$$\frac{\partial \alpha_D}{\partial \sigma} = \frac{\begin{vmatrix} -\lambda_{\sigma} & \lambda_{a_D} \\ -\lambda^*_{\sigma} & \lambda^*_{a_D} \end{vmatrix}}{\begin{vmatrix} \lambda_{\alpha_D} & \lambda_{a_D} \\ \lambda^*_{\alpha_D} & \lambda^*_{a_D} \end{vmatrix}} \to \frac{-L(\tau_S(\alpha_D))\left[u(t_S(a_D)) - u(t_D(a_D))\right]}{-\left[u(t_S(a_D)) - u(t_D(a_D))\right] \times \left[u(\tau_S(\alpha_D)) - u(\tau_D(\alpha_D))\right]} < 0$$